A Total-Variation-Diminishing Finite-Difference Scheme for the Transient Response of a Lossless Transmission Line

Kyu-Pyung Hwang and Jian-Ming Jin

***Abstract*—Total-variation-diminishing (TVD) finite-difference schemes have been used in computational fluid dynamics for accurate solutions of fluid problems involving shock phenomena. This paper investigates the possibility of their application in transient electromagnetic-wave problems. A lossless transmission line with a resistive load is considered to illustrate the application. A TVD Lax–Wendroff finite-difference scheme is presented for the numerical solution of transmission-line equations in time domain. Numerical results show that the TVD scheme can approximate the discontinuous waveforms with remarkable accuracy.**

***Index Terms*—Finite-difference method, transmission line, TVD algorithm.**

I. Introduction

The concept of a total-variation-diminishing (TVD) algorithm has been developed in the computational fluid dynamics (CFD's) community during the 1980's for numerical finite-difference solutions of fluid problems involving shock phenomena [1]. A variety of numerical schemes with the TVD property have been successfully used in CFD's to find solutions containing discontinuities with remarkable accuracy. In this paper, the possibility of using the TVD algorithm as a numerical technique is investigated for simulating transient electromagnetic phenomena in computational electromagnetics. Transients involving discontinuous waveforms can be readily found in terminated transmission-line problems where a step function or a rectangular pulse is applied at the input terminal by a voltage source. The transient response of a transmission line has been obtained either as a sum of multiple reflections or as a sum of a theoretically infinite number of residue terms [2], [3]. This paper proposes a TVD finite-difference scheme for calculating the transient response of a lossless transmission line terminated with a resistive load. The scheme uses the Lax–Wendroff discretization as the basic time integration method and adds nonlinear discretization terms effecting TVD property in each time step to control the numerically generated oscillations. These oscillations are nonphysical errors occurred in discontinuous numerical solutions because the finite-difference approximation based on the Taylor series expansion is not valid for discontinuous functions. The TVD Lax–Wendroff scheme is applied to the propagation of discontinuous pulses on a lossless transmission line with a resistive load to demonstrate the capability of the TVD algorithm to eliminate spurious oscillations.

II. Formulation

To gain a basic concept of the TVD algorithm, consider a scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0. \tag{1}$$

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The authors are with the Center for Computational Electromagnetics, Department of Electrical and Computer Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801-2991 USA.

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The total variation (TV) of any physically admissible solution

$$\mathrm{TV} = \int \left| \frac{\partial u}{\partial x} \right| dx \tag{2}$$

does not increase in time [4]. The TV in x of a discrete solution to a scalar conservation law is defined by

$$\mathrm{TV}\,(u) = \sum_i |u_{i+1} - u_i|. \tag{3}$$

A numerical solution is said to be of bounded TV or TV stable if the TV is uniformly bounded in t and Δx. A numerical scheme is said to be TVD if

$$\mathrm{TV}\,(u^{n+1}) \le \mathrm{TV}\,(u^n) \tag{4}$$

where n denotes the time level. Davis [5] showed that the TVD finite-difference scheme, which was analyzed by Sweby [6], can be interpreted as a Lax–Wendroff scheme plus an upwind weighted artificial dissipation term. The TVD finite-difference scheme used in this paper to solve transmission-line equations in time domain follows Davis's approach in that the TVD artificial dissipation terms are separated from the basic Lax–Wendroff discretization terms.

General transmission-line equations are written as

$$L\frac{\partial i}{\partial t} + \frac{\partial v}{\partial z} + Ri = 0 \tag{5}$$

$$C\frac{\partial v}{\partial t} + \frac{\partial i}{\partial z} + Gv = 0 \tag{6}$$

where

- R resistance per unit length (Ω/m);
- L inductance per unit length (H/m);
- G conductance per unit length (S/m);
- C capacitance per unit length (F/m).

In this paper, we assume that $R = 0$, $G = 0$, and L and C are constants for simplicity. To apply the TVD Lax–Wendroff finite-difference scheme to the above equations, we may cast them in conservation law form as follows:

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial z} = \mathbf{0} \tag{7}$$

where

$$\boldsymbol{U} = \begin{bmatrix} i \\ v \end{bmatrix} \quad \boldsymbol{F} = \begin{bmatrix} v/L \\ i/C \end{bmatrix}. \tag{8}$$

Since L and C are constants, (7) can be rewritten as

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{A}\frac{\partial \boldsymbol{U}}{\partial z} = \mathbf{0} \tag{9}$$

where

$$\boldsymbol{U} = \begin{bmatrix} i \\ v \end{bmatrix} \quad \boldsymbol{A} = \begin{bmatrix} 0 & 1/L \\ 1/C & 0 \end{bmatrix}. \tag{10}$$

The matrix $\boldsymbol{A}$ is the constant coefficient system matrix for the given transmission-line equations. Its eigenvalues are readily found as

$$\lambda^- = -\frac{1}{\sqrt{LC}} \quad \lambda^+ = \frac{1}{\sqrt{LC}} \tag{11}$$

which imply that the given system of partial differential equations is hyperbolic. The matrix $\boldsymbol{P}$ and its inverse $\boldsymbol{P}^{-1}$ which diagonalize $\boldsymbol{A}$ can be found as

$$\boldsymbol{P} = \begin{bmatrix} \sqrt{C} & \sqrt{C} \\ -\sqrt{L} & \sqrt{L} \end{bmatrix} \quad \boldsymbol{P}^{-1} = \begin{bmatrix} 1/2\sqrt{C} & -1/2\sqrt{L} \\ 1/2\sqrt{C} & 1/2\sqrt{L} \end{bmatrix} \tag{12}$$

which lead to

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{bmatrix} -1/\sqrt{LC} & 0 \\ 0 & 1/\sqrt{LC} \end{bmatrix} = \begin{bmatrix} \lambda^- & 0 \\ 0 & \lambda^+ \end{bmatrix}. \tag{13}$$

We can use $\boldsymbol{P}^{-1}$ to obtain the Riemann invariants or the characteristic variables which remain constant along the two characteristic curves associated with λ^- and λ^+ in the given system of equations. That is,

$$\begin{aligned}\boldsymbol{Q} &= \boldsymbol{P}^{-1}\boldsymbol{U} \\ &= \frac{1}{2}\begin{bmatrix} 1/\sqrt{C} & -1/\sqrt{L} \\ 1/\sqrt{C} & 1/\sqrt{L} \end{bmatrix}\begin{bmatrix} i \\ v \end{bmatrix} \\ &= \begin{bmatrix} i/2\sqrt{C} & -v/2\sqrt{L} \\ i/2\sqrt{C} & v/2\sqrt{L} \end{bmatrix} \\ &= \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}\end{aligned} \tag{14}$$

where q_1 and q_2 are the Riemann invariants associated with waves moving in the $-z$- and the $+z$-directions, respectively.

Following Davis's approach, the system of partial differential equations for a lossless transmission line can be discretized as

$$\begin{aligned}\boldsymbol{U}_k^{n+1} =& \boldsymbol{U}_k^n - \frac{1}{2}\left(\frac{\Delta t}{\Delta z}\right)\boldsymbol{A}\left(\boldsymbol{U}_{k+1}^n - \boldsymbol{U}_{k-1}^n\right) \\ &+ \frac{1}{2}\left(\frac{\Delta t}{\Delta z}\right)^2\boldsymbol{A}^2\left(\boldsymbol{U}_{k+1}^n - 2\boldsymbol{U}_k^n + \boldsymbol{U}_{k-1}^n\right) \\ &+ \boldsymbol{P}\boldsymbol{D}_{k+1/2}^n\left(\boldsymbol{Q}_{k+1}^n - \boldsymbol{Q}_k^n\right) - \boldsymbol{P}\boldsymbol{D}_{k-1/2}^n\left(\boldsymbol{Q}_k^n - \boldsymbol{Q}_{k-1}^n\right)\end{aligned} \tag{15}$$

where $z = k\Delta z$, $t = n\Delta t$. Except for the last two terms involving the variations of the Riemann invariants, the discretization is equivalent to the Lax–Wendroff finite-difference scheme for the given system of equations. In (15),

$$\boldsymbol{D}_{k+1/2}^n = D_{k+1/2}^n\boldsymbol{I} \quad \boldsymbol{D}_{k-1/2}^n = D_{k-1/2}^n\boldsymbol{I} \tag{16}$$

where $\boldsymbol{I}$ denotes the identity matrix and

$$\begin{aligned}D_{k+1/2}^n &= D_{k+1/2}^{+n}\left(r_k^{+n}\right) + D_{k+1/2}^{-n}\left(r_{k+1}^{-n}\right) \\ &= \frac{\nu^+}{2}\left(1-\nu^+\right)\left[1-\phi\left(r_k^{+n}\right)\right] + \frac{\nu^-}{2}\left(1+\nu^-\right) \\ &\quad\cdot\left[\phi\left(r_{k+1}^{-n}\right)-1\right]\end{aligned} \tag{17}$$

$$\begin{aligned}D_{k-1/2}^n &= D_{k-1/2}^{+n}\left(r_{k-1}^{+n}\right) + D_{k+1/2}^{-n}\left(r_k^{-n}\right) \\ &= \frac{\nu^+}{2}\left(1-\nu^+\right)\left[1-\phi\left(r_{k-1}^{+n}\right)\right] + \frac{\nu^-}{2}\left(1+\nu^-\right) \\ &\quad\cdot\left[\phi\left(r_k^{-n}\right)-1\right]\end{aligned} \tag{18}$$

where $\nu^+ = \lambda^+\Delta t/\Delta z$, $\nu^- = \lambda^-\Delta t/\Delta z$. In this paper, the superscripts "+" and "−" denote the associated wave propagation directions. Here, $D_{k+1/2}^{+n}$ acts only on the waves propagating in the $+z$-direction and $D_{k+1/2}^{-n}$ acts only on the waves propagating in the $-z$-direction. r_k^{+n}, r_{k+1}^{-n}, r_{k-1}^{+n}, and r_k^{-n} will be given later as the ratios of consecutive variations of the Riemann invariants computed at the time level n. ϕ is called the flux limiter, which controls the additional numerical contributions in a nonlinear fashion. There exist several types of flux limiter functions used in CFD's [1]. In this paper, the Superbee limiter proposed by Roe [7] is used for its excellent resolution property in sharp discontinuities. It is defined as

$$\phi(r) = \max\left[0, \min(2r, 1), \min(r, 2)\right]. \tag{19}$$

In this TVD Lax–Wendroff scheme for transmission-line equations, the dual sets of computations are performed for waves propagating in both directions. That is,

$$\boldsymbol{U} = \boldsymbol{U}^+ + \boldsymbol{U}^- = \begin{bmatrix} i^+ \\ v^+ \end{bmatrix} + \begin{bmatrix} i^- \\ v^- \end{bmatrix} \tag{20}$$

and $\boldsymbol{U}^+$ and $\boldsymbol{U}^-$ are computed separately in each time step. This is for an easy and accurate implementation of the boundary condition at both input and output terminals. First, consider $\boldsymbol{U}^+ = [i^+, v^+]^T$. From (15), the discretized equations for two components can be written as

$$\begin{aligned}i_k^{+n+1} =& i_k^{+n} - \frac{1}{2}\left(\frac{\Delta t}{\Delta z}\right)\frac{1}{L}\left(v_{k+1}^{+n} - v_{k-1}^{+n}\right) \\ &+ \frac{1}{2}\left(\frac{\Delta t}{\Delta z}\right)^2\frac{1}{LC}\left(i_{k+1}^{+n} - 2i_k^{+n} + i_{k-1}^{+n}\right) + t_1^{+n}\end{aligned} \tag{21}$$

$$\begin{aligned}v_k^{+n+1} =& v_k^{+n} - \frac{1}{2}\left(\frac{\Delta t}{\Delta z}\right)\frac{1}{C}\left(i_{k+1}^{+n} - i_{k-1}^{+n}\right) \\ &+ \frac{1}{2}\left(\frac{\Delta t}{\Delta z}\right)^2\frac{1}{LC}\left(v_{k+1}^{+n} - 2v_k^{+n} + v_{k-1}^{+n}\right) + t_2^{+n}\end{aligned} \tag{22}$$

where t_1^{+n} and t_2^{+n} denote the TVD artificial dissipation terms which produce a correcting numerical flux for i_k^{+n+1} and v_k^{+n+1}, respectively. Using the TVD term expression in (15), we obtain

$$\begin{aligned}\begin{bmatrix} t_1^{+n} \\ t_2^{+n} \end{bmatrix} &= D_{k+1/2}^n\boldsymbol{P}\left(\boldsymbol{Q}_{k+1}^n - \boldsymbol{Q}_k^n\right) - D_{k-1/2}^n\boldsymbol{P}\left(\boldsymbol{Q}_k^n - \boldsymbol{Q}_{k-1}^n\right) \\ &= D_{k+1/2}^n\begin{bmatrix} \sqrt{C}\left(q_{1\,k+1}^{+n} - q_{1\,k}^{+n}\right) + \sqrt{C}\left(q_{2\,k+1}^{+n} - q_{2\,k}^{+n}\right) \\ -\sqrt{L}\left(q_{1\,k+1}^{+n} - q_{1\,k}^{+n}\right) + \sqrt{L}\left(q_{2\,k+1}^{+n} - q_{2\,k}^{+n}\right) \end{bmatrix} \\ &\quad - D_{k-1/2}^n\begin{bmatrix} \sqrt{C}\left(q_{1\,k}^{+n} - q_{1\,k-1}^{+n}\right) + \sqrt{C}\left(q_{2\,k}^{+n} - q_{2\,k-1}^{+n}\right) \\ -\sqrt{L}\left(q_{1\,k}^{+n} - q_{1\,k-1}^{+n}\right) + \sqrt{L}\left(q_{2\,k}^{+n} - q_{2\,k-1}^{+n}\right) \end{bmatrix}\end{aligned} \tag{23}$$

where

$$\begin{bmatrix} q_{1\,k}^{+n} \\ q_{2\,k}^{+n} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} i_k^{+n}/\sqrt{C} - v_k^{+n}/\sqrt{L} \\ i_k^{+n}/\sqrt{C} + v_k^{+n}/\sqrt{L} \end{bmatrix}. \tag{24}$$

Note that $q_1^+ = 0$ in this case because it is the Riemann invariant associated with waves propagating in the $-z$-direction. Therefore, (23) reduces to

$$\begin{bmatrix} t_1^{+n} \\ t_2^{+n} \end{bmatrix} = \begin{bmatrix} D_{k+1/2}^{+n}\sqrt{C}\left(q_{2\,k+1}^{+n} - q_{2\,k}^{+n}\right) \\ D_{k+1/2}^{+n}\sqrt{L}\left(q_{2\,k+1}^{+n} - q_{2\,k}^{+n}\right) \end{bmatrix} - \begin{bmatrix} D_{k-1/2}^{+n}\sqrt{C}\left(q_{2\,k}^{+n} - q_{2\,k-1}^{+n}\right) \\ D_{k-1/2}^{+n}\sqrt{L}\left(q_{2\,k}^{+n} - q_{2\,k-1}^{+n}\right) \end{bmatrix}. \tag{25}$$

Using (17) and (18), we have the discretized equations for the TVD artificial dissipation terms as follows:

$$\begin{aligned}t_1^{+n} =& \sqrt{C}\frac{\nu^+}{2}\left(1-\nu^+\right)\left[1-\phi\left(r_k^{+n}\right)\right]\left(q_{2\,k+1}^{+n} - q_{2\,k}^{+n}\right) \\ &- \sqrt{C}\frac{\nu^+}{2}\left(1-\nu^+\right)\left[1-\phi\left(r_{k-1}^{+n}\right)\right]\left(q_{2\,k}^{+n} - q_{2\,k-1}^{+n}\right)\end{aligned} \tag{26}$$

$$\begin{aligned}t_2^{+n} =& \sqrt{L}\frac{\nu^+}{2}\left(1-\nu^+\right)\left[1-\phi(r_k^{+n})\right]\left(q_{2\,k+1}^{+n} - q_{2\,k}^{+n}\right) \\ &- \sqrt{L}\frac{\nu^+}{2}\left(1-\nu^+\right)\left[1-\phi\left(r_{k-1}^{+n}\right)\right]\left(q_{2\,k}^{+n} - q_{2\,k-1}^{+n}\right)\end{aligned} \tag{27}$$

where

$$r_k^{+n} = \frac{q_{2\,k}^{+n} - q_{2\,k-1}^{+n}}{q_{2\,k+1}^{+n} - q_{2\,k}^{+n}} \quad r_{k-1}^{+n} = \frac{q_{2\,k-1}^{+n} - q_{2\,k-2}^{+n}}{q_{2\,k}^{+n} - q_{2\,k-1}^{+n}}. \tag{28}$$

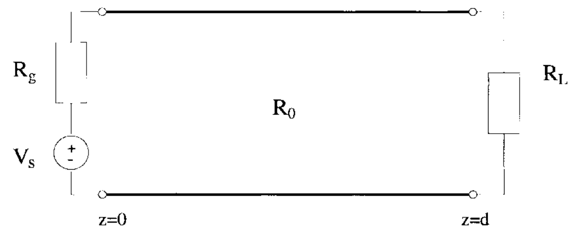


Fig. 1. A lossless transmission line with a resistive load and $d = 400$ m.

For $\boldsymbol{U}^- = [i^-, v^-]^T$, a similar procedure applies. The resulting discretized equations can be found as

$$\begin{aligned} t_1^{-n} &= \sqrt{C}\frac{\nu^-}{2}\left(1+\nu^-\right)\left[\phi\left(r_{k+1}^{-n}\right)-1\right]\left(q_{1\;k+1}^{-n}-q_{1\;k}^{-n}\right) \\ &\quad - \sqrt{C}\frac{\nu^-}{2}\left(1+\nu^-\right)\left[\phi\left(r_{k}^{-n}\right)-1\right]\left(q_{1\;k}^{-n}-q_{1\;k-1}^{-n}\right) \quad (29) \\ t_2^{-n} &= -\sqrt{L}\frac{\nu^-}{2}\left(1+\nu^-\right)\left[\phi\left(r_{k+1}^{-n}\right)-1\right]\left(q_{1\;k+1}^{-n}-q_{1\;k}^{-n}\right) \\ &\quad + \sqrt{L}\frac{\nu^-}{2}\left(1-\nu^-\right)\left[\phi\left(r_{k}^{-n}\right)-1\right]\left(q_{1\;k}^{-n}-q_{1\;k-1}^{-n}\right) \quad (30) \end{aligned}$$

where

$$r_{k+1}^{-n} = \frac{q_{1\;k+2}^{-n}-q_{1\;k+1}^{-n}}{q_{1\;k+1}^{-n}-q_{1\;k}^{-n}} \quad r_{k}^{-n} = \frac{q_{1\;k+1}^{-n}-q_{1\;k}^{-n}}{q_{1\;k}^{-n}-q_{1\;k-1}^{-n}}. \quad (31)$$

III. Numerical Results

In this section, we present some numerical results to demonstrate the advantages of the TVD scheme. The problem considered is a lossless transmission line with a resistive load, as illustrated in Fig. 1. The internal resistor R_g is connected in series with the voltage source V_s. The characteristic resistance of the line is R_0 and the resistance of the load is R_L. In all the calculations, we set $\Delta z = 1$ m and $\Delta t = 3$ ns.

The propagation of a rectangular voltage pulse is considered first to verify the TVD property of the proposed finite-difference scheme. The voltage waveform is specified by the function of time at the voltage source as

$$V_s(t) = 3[U(t-t_1) - U(t-t_2)] \quad (\mathrm{V}) \quad (32)$$

where $t_1 = 150$ ns, $t_2 = 300$ ns in this case. The results are shown in Fig. 2. The basic Lax–Wendroff numerical solution clearly exhibits the spurious oscillations mainly due to high-frequency numerical-dispersion errors. The TVD Lax–Wendroff numerical solution is free from the spurious errors and resolves the jumps in several grid points, compared with the exact solution, which is obtained by the spatial translation of the voltage waveform incident at the input terminals. The numerical dispersion and dissipation of the Lax–Wendroff finite-difference scheme is dependent upon the Courant number ν, which is given in our case as

$$\nu = \left|\lambda^{\pm}\frac{\Delta t}{\Delta z}\right|. \quad (33)$$

The resolution of the TVD Lax–Wendroff solution around the discontinuities is known to be practically independent of the number of time steps and the Courant number within its stability limit [1]. The stability condition of the Lax–Wendroff scheme requires $\nu \leq 1$. In all the numerical results in this paper, the Courant number of 0.9 is used for practical comparison.

To demonstrate the accuracy of the proposed finite-difference scheme, the computed TVD solutions are compared with the exact solutions which are found using the multiple reflection diagram

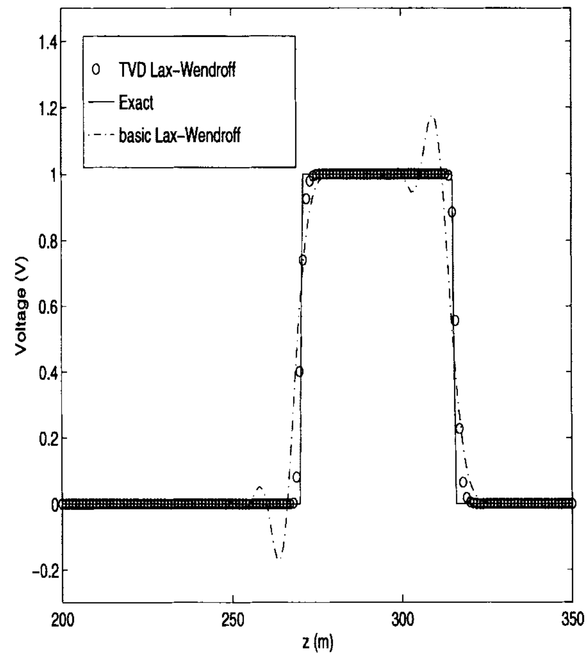
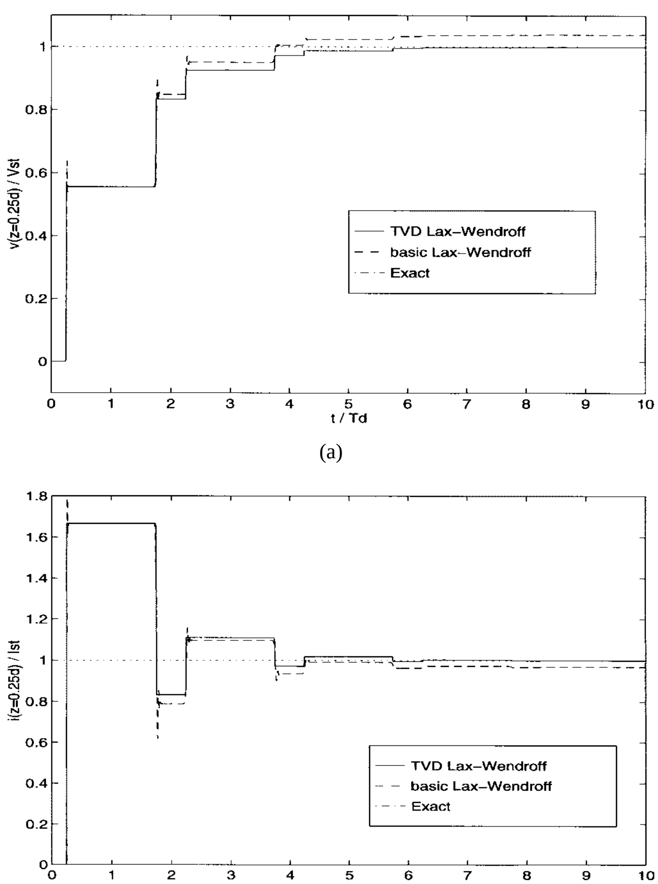


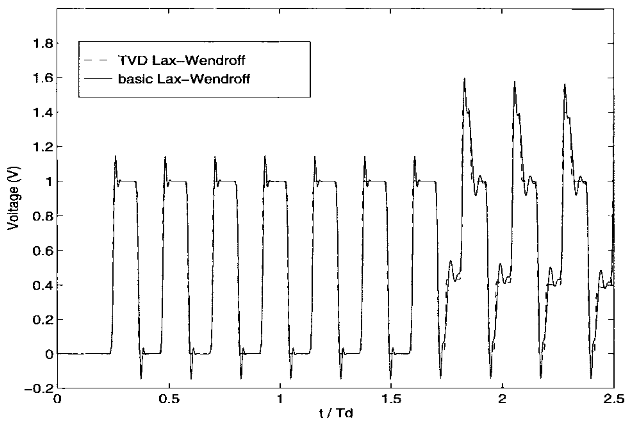
Fig. 2. Comparison of the TVD Lax–Wendroff solution and the basic Lax–Wendroff solution with the exact voltage waveform after 400 time steps plotted in space. $R_L = 3R_0$, $R_g = 2R_0$.



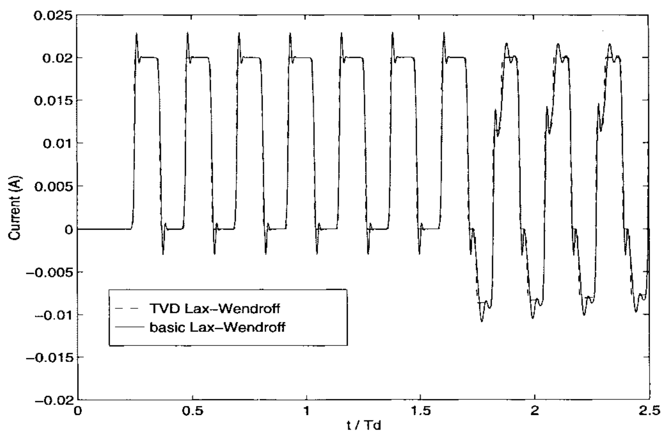
(a)

(b)

Fig. 3. Transient response of the lossless transmission line computed at $z = 0.25d$ versus the normalized time. $R_L = 3R_0$, $R_g = 2R_0$. (a) Voltage. (b) Current. (Note that the TVD Lax–Wendroff solution coincides with the exact solution.)



(a)



(b)

Fig. 4. Transient response of the lossless transmission line with a time-varying resistive load computed at $z = 0.25d$ versus the normalized time. $R_g = R_0$, $T = 300$ ns, $t_w = 150$ ns, $t_0 = 5\,\mu$s. (a) Voltage. (b) Current.

method [3]. The response to a step function of the given transmission-line system is given in Fig. 3. In this case,

$$V_s = 3U(t) \quad (\mathrm{V}) \tag{34}$$

and the normalized voltage and current responses computed at $z = 0.25d$ are plotted versus the normalized time. The normalization factors are defined as

$$T_d = d\sqrt{LC} \quad V_{st} = \frac{R_L}{R_g + R_L}V_s \quad I_{st} = \frac{V_s}{R_g + R_L}. \tag{35}$$

Thus, the voltage and the current converge to their steady-state values V_{st}, I_{st}, respectively, as time goes on. Compared with the exact solution, the transient voltage and current responses computed by the TVD algorithm accurately predict the discontinuous staircase waveforms. However, the results computed by the basic Lax–Wendroff method exhibit spurious oscillations and converge to a final result, which deviates from the exact solution because of error accumulation.

A time-varying load with a pulse-train input is considered next, for which the multiple reflection calculation becomes cumbersome. In this case,

$$V_s(t) = 2[U(t) - U(t_w)] \quad (\mathrm{V}), \qquad 0 < t < T \tag{36}$$

and $V_s(t) = V_s(t+T)$ for all t. The time-varying load is such that its resistance changes exponentially from $3R_0$ to R_0 as follows:

$$R_L(t) = 3R_0 - 2R_0\left[1 - e^{-t/t_0}\right]. \tag{37}$$

Fig. 4 compares the voltage and current responses computed at $z = 0.25d$ using the TVD Lax–Wendroff and the basic Lax–Wendroff finite-difference methods. The comparison shows that the TVD solutions describe the complicated discontinuous waveforms without spurious oscillation. Finally, we note that the computing time using the TVD method is about two times that using the basic Lax–Wendroff method. The time-domain TVD finite-difference approach proposed in this paper could find further applications in time-dependent situations such as the example considered here.

IV. Conclusion

In this paper, the TVD algorithm has been introduced for the finite-difference solution of transient electromagnetic-wave problems. To demonstrate its application, the Lax–Wendroff finite-difference scheme has been adapted with the TVD enhancement for the numerical solution of the transient response of a lossless transmission-line system. The TVD Lax–Wendroff solutions have demonstrated a remarkable accuracy in predicting transient waveforms consisting of numerous discontinuities. Further work is being carried out for more challenging applications in electromagnetics.

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